Imprecision Attenuates Updating*

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Abstract

Agents often base decisions on noisy signals, attenuating Bayesian updating toward the prior expectation – a well-established phenomenon in the normal-normal signal-extraction model. We show this attenuation effect extends to all symmetric, log-concave distributions. By introducing a notion of precision based on likelihood-ratio dominance, we prove that when both the prior and noise are symmetric and log-concave, the posterior mean moves closer to the prior mean as the signal becomes less precise. We discuss applications to cognitive imprecision, prior precision, and overconfidence.

Keywords: signal extraction, Bayes's rule, reversion to the mean, cognitive imprecision
JEL Classifications: C11, D83, D84

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1 Introduction

In many economic contexts, agents often do not respond optimally to fundamental variables: numerical estimates are biased toward default values (Tversky and Kahneman, 1974), firms and households adjust only partially to changes in macroeconomic conditions (Sims, 2003), and consumers underreact to non-salient taxes (Chetty et al., 2009). A recent body of literature suggests that these and similar behavioral phenomena can be explained by *cognitive imprecision* (Gabaix, 2019; Woodford, 2020; Enke and Graeber, 2023; Enke et al., 2024): agents base their decisions on noisy internal signals of the true variables of interest.

Models of cognitive imprecision account for such behavior by showing noise in cognition attenuates the Bayesian updating process, thereby compressing behavior towards some default action. This effect is typically formalized through the *normal-normal model*: the agent observes a noisy signal $S = X + \varepsilon$, where the state $X \in \mathbb{R}$ is normally distributed, and ε is independent, normally distributed noise. The agent's posterior mean is then compressed toward the prior mean, more so when the signal is less precise: the posterior mean lies between the signal and the prior mean and the posterior mean is closer to the prior mean the larger the variance of ε.[1] We refer to this effect as *imprecision attenuates updating*. When the agent's action is determined by their posterior mean, then attenuation of updating translates to attenuation of behavior. One important piece of evidence for cognitive imprecision relies on this effect: subjects who report higher cognitive uncertainty tend to exhibit more attenuated behavioral responses (Enke and Graeber, 2023; Enke et al., 2024).

Despite the empirical relevance of this attenuation effect, it remained unknown how far it extends beyond normal distributions, which rely on strong parametric assumptions. Although normal distributions are justified in certain contexts — such as through the central limit theorem or under rational inattention with a normal prior and quadratic loss — these justifications are limited to specific situations. Therefore, identifying a non-parametric class of signal structures under which imprecision attenuates updating is desirable to provide a more robust theoretical foundation for interpreting empirical observations as implications of cognitive imprecision.

To address this gap, we show imprecision attenuates updating holds for all additive noise models with symmetric, log-concave distributions: when the state X and noise ε have (possibly different) symmetric and log-concave densities, the posterior mean moves closer to the prior mean as the signal becomes less precise, for any signal realization s (Theorem 1).

To formulate this result, we introduce an appropriate notion of precision, which we call the *precision order*. It requires the noise distribution of a less precise signal to be further away from zero in the sense of likelihood-ratio domination: $\tilde{S} = X + \tilde{\varepsilon}$ is less precise than $S = X + \varepsilon$ if the likelihood ratio $f_{\tilde{\varepsilon}}(x)/f_\varepsilon(x)$ is weakly increasing as x moves away from 0. More concretely, we show that if we scale up a log-concave error, the resulting signal is less precise in our sense, and thus, our precision

[1] Formally, if $X \sim \mathcal{N}(\mu, \sigma_X^2)$ and $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, then

$$\mathbb{E}[X|S = s] = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_\varepsilon^2} s + \left(1 - \frac{\sigma_X^2}{\sigma_X^2 + \sigma_\varepsilon^2}\right) \mu.$$

The larger σ_ε^2, the closer the posterior mean is to the prior mean.

order applies to log-concave location-scale experiments.

Symmetric, log-concave distributions include many common probability distributions beyond the normal, such as logistic, extreme value, and double-exponential distributions. Our focus on this class is motivated by Chambers and Healy (2012), who demonstrate symmetry and quasi-concavity (unimodality) are necessary conditions for the posterior mean to robustly lie between the prior mean and the signal realization. More precisely, they show that if the prior is not symmetric and quasi-concave, one can find a symmetric and quasi-concave error (and vice versa) under which the posterior mean does not lie between the prior mean and the signal realization. We show one need only strengthen quasi-concavity to log-concavity to obtain our result.

Although our main result extends the attenuation effect to a broader class of distributions, some caution is warranted. As Chambers and Healy (2012) demonstrate, without symmetry and quasi-concavity, one can easily construct examples where the agent overreacts to signals; that is, the posterior mean is more extreme than the signal. Moreover, we show adding additional independent noise to a signal structure can lead the posterior mean to become more extreme if the prior is not log-concave (Appendix 6.1).

Finally, we leverage Theorem 1 to establish implications for signal extraction problems, which we believe are of broader economic relevance:

- *Prior Precision*: Our main theorem implies a converse comparative-statics result regarding prior precision (Corollaries 2 and 3): increasing the precision of the prior brings the posterior mean closer to the prior mean for any given signal realization.

- *Average Posterior Means*: We extend our analysis to *average* posterior means conditional on the true state, which are relevant when agents acquire conditionally independent signals about a common variable. We show that, conditional on the true state, the average posterior mean lies between the state and the prior mean (Proposition 1).

- *Comparative Statics for Average Posterior Mean*: Greater (over)confidence in the signal brings the average posterior mean closer to the true state (Proposition 2), whereas greater prior precision brings it closer to the prior mean (Proposition 3).

Related Literature We show new theoretical results for signal structures with additive independent noise, also called location experiments. Boll (1955) shows one location experiment is Blackwell-dominated by another if and only if its error is obtained from the other's error by adding another independent error. Lehmann (1988) studies the value of information for location experiments with log-concave errors in monotone decision problems. Kartik et al. (2021) show a comparative-statics result on posterior means for agents with heterogeneous priors under monotone likelihood-ratio experiments, of which location-experiments with log-concave errors are an example. Dawid (1973) shows that, broadly speaking, when the error is more heavy-tailed than the prior, extreme observations get rejected in the sense that the posterior is close to the prior (for a review of the ensuing literature, see O'Hagan and Pericchi, 2012).

2 Model

We restrict attention to signal structures, where the signal equals the one-dimensional state plus some independent noise, which are called location experiments. Formally, the random signal S is a location experiment if it is the sum of the random state X and an random error ε that is independent of X, which we write as $\varepsilon \perp\!\!\!\perp X$. We assume that the error has mean 0, that is, the signal is unbiased. If not, one could easily derive an unbiased signal by subtracting the mean of ε from S.

Assumption 1 (Location Experiment)**.**

$$S = X + \varepsilon \tag{1}$$

$$\varepsilon \perp\!\!\!\perp X \tag{2}$$

$$\mathbb{E}[\varepsilon] = 0 \tag{3}$$

Further, we make the technical assumption that the state X and error ε admit positive densities f_X and f_ε, and that these densities are absolutely Lebesgue-integrable, which guarantees finite means and finite conditional expectation $\mathbb{E}[X|S = s]$. We call f_X the *prior density* and f_ε the *error density.*

Assumption 2. *X and ε admit positive densities $f_X, f_\varepsilon \in \mathcal{L}^1(\mathbb{R})$.*

Following Chambers and Healy (2012), we assume symmetric and quasi-concave prior and error densities. They show that without these assumptions, the posterior mean does not necessarily lie between the prior mean and the signal realization. We believe it is only interesting to speak of *more* updating toward the signal, when there is updating toward the signal in the first place.

Assumption 3. *The densities f_X and f_ε are symmetric and quasi-concave.*

In particular, that the prior density f_X is symmetric around $\mathbb{E}[X]$ and the error density f_ε is symmetric around 0 by (3).

Chambers and Healy (2012) show the following result, which we include for later reference.

Fact 1 (Chambers and Healy, 2012, Proposition 3)**.** *Under Assumptions 1 to 3, for any signal realization s, the posterior mean $\mathbb{E}[X|S = s]$ lies weakly between the prior mean $\mathbb{E}[X]$ and the signal s. Formally,*

$$\forall s \leq \mathbb{E}[X]: \qquad s \leq \mathbb{E}[X|S = s] \leq \mathbb{E}[X],$$
$$\forall s \geq \mathbb{E}[X]: \qquad s \geq \mathbb{E}[X|S = s] \geq \mathbb{E}[X].$$

Throughout the paper, we maintain Assumptions 1 to 3. In particular, all considered prior and error densities are assumed to be positive, symmetric, and quasi-concave, unless otherwise specified.

3 Precision Order

One challenge to formalizing the idea that imprecision attenuates updating is to find the right order on signal structures. The well known Blackwell order is not appropriate for two reasons. First,

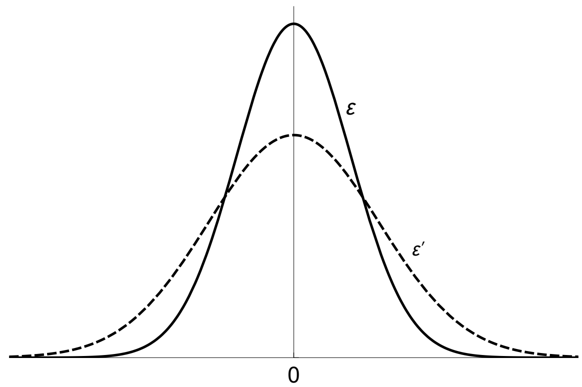

Figure 1: Mean-zero normal distributions with scale parameters $\sigma = 1$ (bold) and $\sigma = 1.5$ (dashed). The former distribution is more precise as the likelihood ratio falls moving away from 0.

we show in Appendix 6.1 that the Blackwell order predicts the *imprecision attenuates updating* effect only under very special conditions. Second, the Blackwell order is very restrictive on location experiments (Boll, 1955; Lehmann, 1988). In particular, the location experiment $S = X + \varepsilon$ with $\varepsilon \sim U[-1, 1]$ does not Blackwell dominate the location experiment $S = X + \varepsilon'$ with $\varepsilon' \sim U[-k, k]$ if k is not an integer, no matter how large k is.

Instead, we introduce a new order, which we call the *precision order* on the class of location experiments that satisfy Assumptions 1 to 3. We first define this order for error variables (or, more generally, for random variables) and then extend it to the associated location experiments. The next section shows that this order delivers the desired result (Theorem 1).

Definition 1 (Precision Order)**.** *Let $\tilde{\varepsilon}$ and ε be random variables with positive, symmetric-around-0, and quasi-concave densities $f_{\tilde{\varepsilon}}$ and f_ε, respectively. We say $\tilde{\varepsilon}$ is less precise than ε if the likelihood ratio*

$$\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}$$

is weakly increasing in x for $x > 0$. Further, we say the location experiment $\tilde{S} = X + \tilde{\varepsilon}$ is less precise than location experiment $S = X + \varepsilon$ if $\tilde{\varepsilon}$ is less precise than ε.

Note that by symmetric of the densities, the definition implies that the likelihood ratio $f_{\tilde{\varepsilon}}(x)(x)/f_\varepsilon(x)$ is weakly *de*creasing for $x < 0$. Thus, the precision order requires that, for positive values, $\tilde{\varepsilon}$ likelihood-ratio dominates ε and, for negative values, $\tilde{\varepsilon}$ is likelihood-ratio dominated by ε. In other words, the less precise location experiment has an error further away from 0 in the sense of likelihood-ratio domination.

Figure 1 shows an example of two normal distributions with different variances, $\varepsilon \sim \mathcal{N}(0, 1)$ and $\varepsilon' \sim \mathcal{N}(0, 1.5^2)$, which are ranked by the precision order. The more precise distribution has a higher density at zero but the density falls faster moving away from zero. More generally, we show that when we *scale up* any symmetric, log-concave density by a constant $k > 1$, then the resulting density is less precise (see section 4.1). Thus, $S = X + \varepsilon$ with $\varepsilon \sim U[-1, 1]$ is more precise than the location experiment $S = X + \varepsilon'$ with $\varepsilon' \sim U[-k, k]$ for all $k > 1$, in contrast to the Blackwell order.[2]

The precision order is at the intersection of two important orders: the convex order, also known as mean-preserving spread order, on the error distributions, and the Lehmann order on location experiments. One can show that if the location experiment $S = X + \varepsilon'$ is less precise than $S = X + \varepsilon$, then the error ε' is a mean-preserving spread of the error ε.[3] The Lehmann order characterizes what experiments are more valuable in all *monotone* decision problems (Lehmann, 1988; Quah and Strulovici, 2009). This order is defined for location experiments only when they have log-concave errors, since these satisfy the monotone likelihood-ratio property required by the Lehmann order. One can show for location experiments with log-concave errors, the precision order implies the Lehmann order.[4]

4 Results

Our main result shows that *imprecision attenuates updating* if we rank location experiments by the precision order, under the additional assumption of a log-concave prior.

Theorem 1 (Imprecision Attenuates Updating)**.** *Let the prior density be log-concave. Under a less precise location experiment, the posterior mean is weakly closer to the prior mean, for any signal realization. Formally, if $\tilde{S} = X + \tilde{\varepsilon}$ is less precise than $S = X + \varepsilon$, then*

$$\begin{aligned} \forall s \leq \mathbb{E}[X]: \qquad & \mathbb{E}[X|S = s] \leq \mathbb{E}[X|\tilde{S} = s] \leq \mathbb{E}[X], \\ \forall s \geq \mathbb{E}[X]: \qquad & \mathbb{E}[X|S = s] \geq \mathbb{E}[X|\tilde{S} = s] \geq \mathbb{E}[X]. \end{aligned}$$

The proof can be found in the Appendix, the idea of which is roughly as follows. Suppose without loss that the signal realization s is larger than the prior mean. One can show that to the right of s, the more precise signal leads to a likelihood-ratio dominated posterior, which decreases the posterior mean, and to the left of the s, the more precise signal leads to a likelihood-ratio dominant posterior, which increases the posterior mean. The crucial part of the proof is to show that the former effect dominates. The proof does this through a sequence of inequalities and bounds, using only elementary properties of Bayesian updating and quasi-concave as well as log-concave functions.

For this as well as the following results, an analogous *strict* version of the result holds. We can instead require that the error is replaced by a *strictly* less precise error in the sense that the likelihood

[2] To be precise, the precision order requires the densities to be positive everywhere. However, we can approximate uniform densities through positive densities.

[3] If S' is less precise than S, then the CDFs of ε' and ε cross once. Together with ε' and ε having equal means, this implies that one distribution majorizes the other, see also Diamond and Stiglitz (1974).

[4] On such location experiments, the Lehmann order coincides with the dispersive order on the error distributions, which requires that any two quantiles are weakly further apart under the more dispersed error distribution. Formally, let ε and ε' have CDFs F and G, respectively, and let F^{-1} and G^{-1} denote here the right-continuous inverses. ε is smaller in the dispersive order than ε' if $\forall\ 0 < \alpha \leq \beta < 1 : F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$. When we truncate the errors to positive values, the precision order implies the likelihood-ratio order. As is known, the likelihood-ratio order implies the hazard rate order (e.g. Shaked and Shanthikumar, 2007, Theorem 1.C.1). The hazard rate order together with densities being log-concave implies the dispersive order (Bagai and Kochar, 1986, Theorem 1). It can be easily seen that if the dispersive order holds for all $1/2 \leq \alpha \leq \beta < 1$, then it holds for all $0 < \alpha \leq \beta < 1$.

ratio in Definition 1 is strictly decreasing. Then, for signal realizations distinct from the prior mean, the posterior mean is strictly closer to s.

To illustrate an immediate implication of the result, suppose two agents observe the same signal but they differ in their assessment of the signal structure. One agent is overconfident in the signal source and updates as if the error was more precise. Then, that agent's posterior mean will be closer to the signal, for any signal realization. In section 4.3, we elaborate on the overconfidence application.

Next, we show that our precision order arises endogenously in *location-scale experiments* with log-concave, and certain non-log-concave, error densities.

4.1 Location-Scale Experiments

We introduce to our location experiment a *scale parameter* $\sigma \in \mathbb{R}_{\geq 0}$ that scales the error term, such that

$$S_\sigma = X + \sigma\varepsilon. \tag{4}$$

We show comparative statics on the scale parameter σ.

Lemma 1. *If ε is symmetric around 0 and $\log f_\varepsilon(e^x)$ is concave, then $\forall \sigma' > \sigma > 0$, $\sigma'\varepsilon$ is less precise than $\sigma\varepsilon$. In particular, $\log f_\varepsilon(e^x)$ is concave if the density $f_\varepsilon(x)$ is log-concave.*[5]

In the Appendix, we show that Lemma 1 follows from a known result. Many commonly used distributions are symmetric and log-concave, such as the normal, logistic, extreme value, and double-exponential distributions. Further, we give examples of symmetric and non-log-concave distributions for which $\log f_\varepsilon(e^x)$ is nevertheless concave, such as the Student-t, Cauchy, and the "double" Pareto distribution.

Together, Theorem 1 and Lemma 1 imply the following important result. By Lemma 1, the result still holds if we weaken the assumption that the error density f_ε is log-concave to $\log f_\varepsilon(e^x)$ being concave.

Corollary 1. *Let the prior and error densities be log-concave. The posterior mean is weakly closer to the prior mean under a larger scale parameter, for any signal realization s. Formally, if $\tilde{\sigma} > \sigma > 0$, then*

$$\begin{aligned} \forall s \leq \mathbb{E}[X]: &\qquad \mathbb{E}[X|S_\sigma = s] \leq \mathbb{E}[X|S_{\tilde{\sigma}} = s] \leq \mathbb{E}[X], \\ \forall s \geq \mathbb{E}[X]: &\qquad \mathbb{E}[X|S_\sigma = s] \geq \mathbb{E}[X|S_{\tilde{\sigma}} = s] \geq \mathbb{E}[X]. \end{aligned}$$

Location-scale experiments with log-concave error density further satisfy the monotone likelihood-ratio property, which implies that the posterior mean is non-decreasing in the signal, as in the normal-normal model. This suggests that location-scale experiments with symmetric and log-concave prior

[5] This can be seen easily if the log-density is differentiable. Define $\phi = \log f$ and $\psi = \exp$. Then, ϕ is concave and decreasing for positive values and ψ is convex, increasing, and obtains positive values only, which implies $(\phi \circ \psi)''(x) = \psi''(x)(\phi' \circ \psi)(x) + (\psi'(x))^2(\phi'' \circ \psi)(x) \leq 0$.

and error density are a useful class of models that maintains key properties of the normal-normal model.

The next section shows comparative statics result for changing the prior instead of the error, exploiting a symmetry in location experiments.

4.2 Comparative Statics on the Prior

For location experiments, the posterior density is symmetric in the prior and in the error density. This follows from the more general property of Bayesian updating that the posterior is proportional to the product of the prior and the likelihood, $p(x|s) \propto p(x,s) = p(x)p(s|x)$. In the case of location experiments, this implies the density of the posterior conditional on $S = s$ is proportional to $f_X(x)f_\varepsilon(s-x)$. Using this insight, Theorem 1 immediately implies a dual result for making the prior more precise.

Corollary 2. *Let the error density be log-concave. The posterior mean is weakly closer to the prior mean under a more precise prior, for any signal realization s. Formally, if* $\mathbb{E}[X] = \mathbb{E}[\tilde{X}]$ *and* $\tilde{X}$ *is more precise than* X*, then*

$$\forall s \leq \mathbb{E}[X]: \qquad \mathbb{E}[X|X+\varepsilon = s] \leq \mathbb{E}[\tilde{X}|\tilde{X}+\varepsilon = s] \leq \mathbb{E}[X],$$
$$\forall s \geq \mathbb{E}[X]: \qquad \mathbb{E}[X|X+\varepsilon = s] \geq \mathbb{E}[\tilde{X}|\tilde{X}+\varepsilon = s] \geq \mathbb{E}[X].$$

To illustrate Corollary 2, suppose that two agents observe the same signal about the state but one agent has a more precise prior (but with the same mean). Corollary 2 implies that the agent with the more precise prior has a posterior mean that is closer to the prior mean, for any signal realization.

By the same argument, an analogous dual result to Corollary 1 holds for scaling the prior instead of the error density. If X has density $f_X(x)$, then kX has density $1/kf_X(x/k)$.

Corollary 3. *Let the error and prior density be log-concave. For any signal realization s, the posterior mean is weakly closer to the prior mean if we scale down the prior. Formally, if* $0 < \tilde{k} < k$ *and we normalize the prior mean to zero,* $\mathbb{E}[X] = 0$*, then*

$$\forall s \leq \mathbb{E}[X]: \qquad \mathbb{E}[kX|kX+\varepsilon = s] \leq \mathbb{E}[\tilde{k}X|\tilde{k}X+\varepsilon = s] \leq \mathbb{E}[X],$$
$$\forall s \geq \mathbb{E}[X]: \qquad \mathbb{E}[kX|kX+\varepsilon = s] \geq \mathbb{E}[\tilde{k}X|\tilde{k}X+\varepsilon = s] \geq \mathbb{E}[X].$$

4.3 Average Posterior Means

Our previous comparative statics results hold for any signal realization and thus speaks to situations where agents observe the same signal. However, in many situations, agents observe *distinct* signals from the same signal structure, that is their signal realization are independent conditional on the state. Further, often we do not observe agent's signal realizations but only the *average* posterior mean, such as when we observe only aggregate behavior from a population of individuals. What can

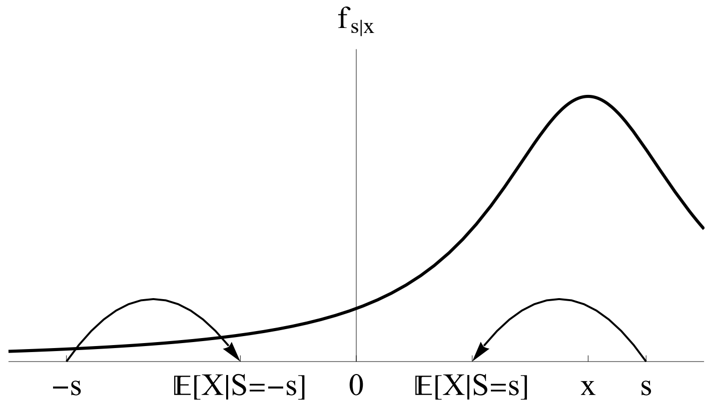

Figure 2: Illustrating the proof of Proposition 1.

be said about comparative statics with respect to the average posterior means given some true state X?

Before we prove comparative statics results, we show that the *average* posterior mean necessarily lies between the state X and the prior mean $\mathbb{E}[X]$, extending Fact 1.

Proposition 1. *For any state x, the conditional average posterior mean $\mathbb{E}[\mathbb{E}[X|S]|X = x]$ lies weakly between the state x and the prior mean $\mathbb{E}[X]$. Formally,*

$$\forall x \leq \mathbb{E}[X]: \qquad x \leq \mathbb{E}[\mathbb{E}[X|S]|X = x] \leq \mathbb{E}[X],$$
$$\forall x \geq \mathbb{E}[X]: \qquad x \geq \mathbb{E}[\mathbb{E}[X|S]|X = x] \geq \mathbb{E}[X].$$

The proof is illustrated using Figure 2. Let $X > \mathbb{E}[X] = 0$. Conditional on state X, the distribution of the signal S is is symmetric around X so its expectation equals X. Taking the conditional expectation $\mathbb{E}[X|S]$ moves the distribution closer to $\mathbb{E}[X]$ as indicated by the arrows. Given our assumptions, the conditional expectation is antisymmetric and $s > 0$ has a higher likelihood than $-s < 0$. Thus, the overall effect on the average posterior mean is negative and $\mathbb{E}[\mathbb{E}[X|S]|X] < \mathbb{E}[S|X] = X$. Further, because the density of s is larger than the density of $-s$, integrating over all s leads to a positive expectation, so $\mathbb{E}[X] < \mathbb{E}[\mathbb{E}[X|S]|X]$.

Overconfidence First, we prove a comparative statics result for overconfidence in the signal. We consider two agents, A and B, that face the same objective signal structure $S = X + \varepsilon$ but update differently because they have different confidence in the signal. That is, agent $i \in \{A, B\}$ forms their conditional expectation $\mathbb{E}_i[X|S = s]$ as if $S = X + \varepsilon_i$, $\mathbb{E}_i[X|S = s] := \mathbb{E}[X|X + \varepsilon_i = s]$. Especially empirically relevant is the case *over*confidence in the signal, also called overprecision, which is pervasive (Moore and Healy, 2008). We define a relative notion of overconfidence by generalizing the definition in Ortoleva and Snowberg (2015), which is based on the normal-normal model, using our Definition 1.

Definition 2. *A is more confident than B in the signal if ε_A is more precise than ε_B.*

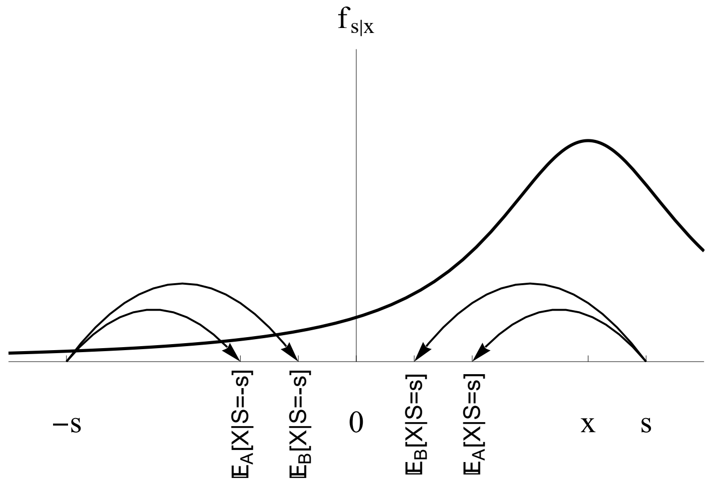

Figure 3: Illustrating the proof of Proposition 2

Using this definition, we prove the following comparative statics result.

Proposition 2. *Let the prior density be log-concave and A be more confident than B. Conditional on any state x, the average posterior mean of A is weakly closer to the state than the average posterior mean of B. Formally,*

$$\forall x \leq \mathbb{E}[X]: \qquad x \leq \mathbb{E}[\mathbb{E}_A[X|S]|X = x] \leq \mathbb{E}[\mathbb{E}_B[X|S]|X = x],$$
$$\forall x \geq \mathbb{E}[X]: \qquad x \geq \mathbb{E}[\mathbb{E}_A[X|S]|X = x] \geq \mathbb{E}[\mathbb{E}_B[X|S]|X = x].$$

This shows that more overconfident agents have, on average, posterior means closer to the state X and further away from their prior mean $\mathbb{E}[X]$.

The proof builds on the proof of Proposition 1 and is illustrated using Figure 3. Because A is more overconfident in the signal, their posterior mean is closer to the signal for any signal realization s as well as $-s$. For positive X, because $s > 0$ is more likely than $-s$, the overall effect on the average posterior mean is positive. Then, the result follows from Proposition 1.

Prior Precision Second, we show comparative statics with respect to the prior precision. Given some state X, consider two agents, A and B, with symmetric and quasi-concave priors that have the same mean, where agent A's prior is more precise than B's. Then, we have from the argument in Proposition 2 and the Corollary 2, we immediately obtain the following result.

Proposition 3. *Let the error density be log-concave and A's prior be less precise than B's. Conditional on any state x, the average posterior mean of A is weakly closer to the state than the average posterior mean of B. Formally,*

$$\forall x \leq \mathbb{E}[X]: \qquad x \leq \mathbb{E}[\mathbb{E}_A[X|S]|X = x] \leq \mathbb{E}[\mathbb{E}_B[X|S]|X = x],$$
$$\forall x \geq \mathbb{E}[X]: \qquad x \geq \mathbb{E}[\mathbb{E}_A[X|S]|X = x] \geq \mathbb{E}[\mathbb{E}_B[X|S]|X = x].$$

5 Conclusion

In this paper, we have extended the attenuation effect of cognitive imprecision beyond the normal-normal model to encompass all symmetric, log-concave distributions. By introducing a new order of precision, based on likelihood-ratio dominance, we demonstrated that imprecision attenuates Bayesian updating toward prior beliefs across a broad class of distributions commonly used in economic modeling. This generalization provides a more robust theoretical foundation for interpreting empirical observations of attenuated behavior as resulting from cognitive imprecision.

Our findings also have broader implications for signal extraction problems. We established comparative statics results regarding prior precision, showing that increased prior precision brings the posterior mean closer to the prior mean for any given signal realization. Additionally, we analyzed average posterior means, demonstrating how overconfidence and prior precision affect the average posterior mean. Perhaps surprisingly, our results show that the posterior means of overconfident agents are on average closer to the truth while those of agents with more precise priors are further away from it.

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6 Appendix

6.1 Blackwell-Order

Boll (1955) shows that location experiment $S = X + \varepsilon$ Blackwell dominates another location experiment $\tilde{S} = X + \tilde{\varepsilon}$ if and only if the error $\tilde{\varepsilon}$ is obtained from the error ε by adding another independent error, that is, $\tilde{\varepsilon} = \varepsilon + \varepsilon'$ with $\varepsilon' \perp\!\!\!\perp \varepsilon$.

We show imprecision attenuates updating if location experiment $S = X + \varepsilon$ Blackwell dominates $\tilde{S} = X + \tilde{\varepsilon}$ under two additional assumptions. First, the posterior mean $\mathbb{E}[X|S = s]$ of the Blackwell dominant location experiment is linear in s. This holds if X and ε are normally distributed. Second, we require that additionally to Assumption 3 that also the added error ε' has a positive, symmetric-around-0, and quasi-concave density.

Proposition 4. *Let location experiment $S = X + \varepsilon$ Blackwell dominate $\tilde{S} = X + \tilde{\varepsilon}$, that is, $\tilde{\varepsilon} = \varepsilon + \varepsilon'$ and $(X, \varepsilon, \varepsilon')$ are jointly independent. Further, let the posterior mean $\mathbb{E}[X|S = s]$ be linear in s and ε' have a positive, symmetric-around-0, and quasi-concave density. Then, the posterior mean $\mathbb{E}[X|\tilde{S} = s]$ is weakly closer to the prior mean than $\mathbb{E}[X|S = s]$. Formally,*

$$\forall s \leq \mathbb{E}[X]: \qquad \mathbb{E}[X|S = s] \leq \mathbb{E}[X|\tilde{S} = s] \leq \mathbb{E}[X],$$
$$\forall s \geq \mathbb{E}[X]: \qquad \mathbb{E}[X|S = s] \geq \mathbb{E}[X|\tilde{S} = s] \geq \mathbb{E}[X].$$

Proof. By the law of iterated expectations and because $\tilde{S}$ is uninformative conditional on S, we have

$$\mathbb{E}[X|\tilde{S}=s] = \mathbb{E}[\mathbb{E}[X|S,\tilde{S}=s]|\tilde{S}=s] = \mathbb{E}[\mathbb{E}[X|S]|\tilde{S}=s].$$

Define $e(s) := \mathbb{E}[X|S=s]$. By assumption, $e(s)$ is linear and hence commutes with the expectation operator,

$$\mathbb{E}[\mathbb{E}[X|S]|\tilde{S}=s] = \mathbb{E}[e(S)|\tilde{S}=s] = e(\mathbb{E}[S|\tilde{S}=s]).$$

The two equations together deliver

$$\mathbb{E}[X|\tilde{S}=s] = e(\mathbb{E}[S|\tilde{S}=s]).$$

The random variable $\tilde{S} = S + \varepsilon'$ viewed as a signal of S satisfies the assumptions in Fact 1. The random variable ε' has a symmetric and quasi-concave density by assumption. The random variable $S = X + \varepsilon$ is symmetric because X and ε are symmetric. And S has a quasi-concave density because X and ε have symmetric and quasi-concave densities (Wintner, 1938).

Thus, by Fact 1, $\mathbb{E}[S|\tilde{S}=s]$ lies weakly between s and $\mathbb{E}[S] = \mathbb{E}[X]$. By e being linear (and it must be linearly increasing), this implies that $\mathbb{E}[X|\tilde{S}=s] = e(\mathbb{E}[S|\tilde{S}=s])$ lies between $e(\mathbb{E}[X]) = \mathbb{E}[X]$ and $e(s) = \mathbb{E}[X|S=s]$. □

The proof of Proposition 4 is instructive to construct counterexamples to *imprecision attenuates updating* under the Blackwell order. These counterexamples occur easily once we deviate from the restrictive assumptions of Proposition 4.

Counterexample with asymmetric ε': Even under normal X and ε, which imply that $\mathbb{E}[X|S=s]$ is linear in s, we need ε' to be symmetric and quasi-concave to apply Fact 1. With skewed distributions for ε', one can construct counterexamples to the conclusion of Proposition 4.

In particular, once can easily construct a binary, mean-zero $\tilde{\varepsilon}$ such that $\mathbb{E}[S|\tilde{S}=s]$ lies outside the interval from $\mathbb{E}[X]$ to s. Let $\tilde{\varepsilon}$ take on value 1 with probability p and value $-p/(1-p)$ with probability $1-p$. Suppose, $X \sim \mathcal{N}(0,\sigma_X^2)$ and $\varepsilon \sim \mathcal{N}(0,\sigma_\varepsilon^2)$, so $S = X + \varepsilon \sim \mathcal{N}(0,\sigma_X^2+\sigma_\varepsilon^2)$. Let $s > 0$. As we let p go to 1, the conditional expectation of S conditional on $\tilde{S}$ converges to the degenerate distribution on $s+1$ and since the normal distribution falls exponentially towards zero, the conditional expectation $\mathbb{E}[S|\tilde{S}=s]$ converges to $s+1$, which lies outside the interval from $\mathbb{E}[X]$ to s. While this argument relies on binary $\tilde{\varepsilon}$, one can perform a similar construction using other left-skewed distributions for $\tilde{\varepsilon}$.

Counterexample with heavy-tailed prior: We can find counterexamples with symmetric $\tilde{\varepsilon}$, if we drop the assumption that ε is normal. To construct an example where the Blackwell dominated location experiment $\tilde{S} = S + \varepsilon + \varepsilon'$ leads to less attenuation, we note the importance of the function $e(s) = \mathbb{E}[X|S=s]$ in our proof of Proposition 4. In particular, if $e(s)$ is convex, then by Jensen's inequality, $\mathbb{E}[e(S)|\tilde{S}=s] > e(\mathbb{E}[S|\tilde{S}=s])$. Further, if $e(s)$ is decreasing, then $e(\mathbb{E}[S|\tilde{S}=s]) > e(s)$ even if $\mathbb{E}[S|\tilde{S}=s] < s$. Thus, if $e(s)$ is *decreasing and convex* over the support of $S|\tilde{S}=s$, then

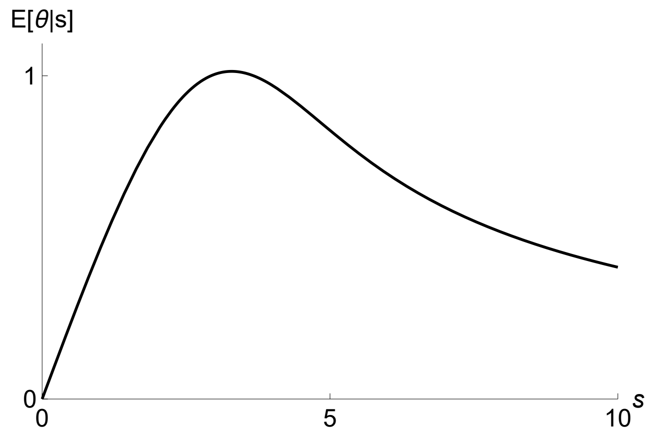

Figure 4: Conditional expectation $\mathbb{E}[X|S = s]$ for normal prior $X \sim \mathcal{N}(0, 1)$ and student t-distributed (with 3 degrees of freedom) error $\varepsilon \sim t_3$.

$\mathbb{E}[X|\tilde{S} = s] = \mathbb{E}[e(S)|\tilde{S} = s] > e(s) = \mathbb{E}[X|S = s]$ and attenuation fails.

How do we generate a (locally) decreasing and convex $e(s)$? This occurs naturally when the error has a heavier tail than the prior, as studied in the literature on Bayesian outlier rejection (O'Hagan and Pericchi, 2012). If the error has a heavier tail than the prior, for large enough signals, the posterior mean decreases until it coincides with the prior mean. This is because large signals are treated as outliers and are attributed purely to the error. For example, when the prior is normally distributed and the error is standard student-t distributed with 3 degrees of freedom (this distribution has finite first and second moments), then the posterior mean takes the form as depicted in Figure 4, which is decreasing and convex in s above a threshold. If we add a symmetric and quasiconcave error with bounded support, we can generate a counterexample to attenuation.

6.2 Proof of Theorem 1

Proof. Without loss of generality, the signal realization is zero, $s = 0$. If $\mathbb{E}[X] = 0$, then the posterior mean is zero by Fact 1 and we are done. Assume $\mathbb{E}[X] < 0$ (the case $\mathbb{E}[X] > 0$ is analogous). We prove that for strictly more precise error, the posterior mean becomes strictly closer to 0.

Let ε denote the error and $\tilde{\varepsilon}$ denote the more precise error and f_ε and $f_{\tilde{\varepsilon}}$ their respective densities. Let f denote the posterior density under error ε after observing signal $s = 0$ and g analogously under error $\tilde{\varepsilon}$.

The posterior mean is

$$\mathbb{E}[X|X + \tilde{\varepsilon} = 0] = \int_{-\infty}^{0} xg(x)dx + \int_{0}^{\infty} xg(x)dx = \int_{0}^{\infty} -x(g(-x) - g(x))dx,$$

and analogously with density f instead of g for the signal with greater noise. The proof revolves around showing that the following integral is positive:

$$\mathbb{E}[X|X + \tilde{\varepsilon} = 0] - \mathbb{E}[X|X + \varepsilon = 0] = \int_{0}^{\infty} -x\big[(g(-x) - g(x)) - (f(-x) - f(x))\big]dx \tag{5}$$

First, we prove the following result regarding the integrand of (5), which uses that $\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}$ is strictly decreasing for $x > 0$.

Lemma 2. *There is some $c > 0$ such that the integrand of (5) is strictly negative for $x \in [0, c)$ and strictly positive for $x \in (-c, \infty)$.*

Proof. Again, the density $f(x)$ is proportional to $f_X(x) f_\varepsilon(x)$ and $g(x)$ to $f_X(x) f_{\tilde{\varepsilon}}(x)$. Thus, there is some factor $C > 0$ such that

$$g(-x) - g(x) = C\frac{f_{\tilde{\varepsilon}}(-x)}{f_\varepsilon(-x)} f(-x) + C\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)} f(x) = C\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}(f(-x) - f(x)),$$

where we have used the symmetry of the error densities. Thus, the integrand of (5) is negative if, and only if, $C\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)} > 1$.

The ratio $\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}$ is strictly decreasing for $x > 0$ by assumption. As both are densities that integrate to 1, the ratio must cross $1/C$ and by strictly decreasing ratio, the crossing point must be unique up to sign. Let c be the unique positive x at which $\frac{g(x)}{f(x)} = 1/C$. Then, for $x \in [0, c)$ we have $C\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)} > 1$ and for $x \in (c, \infty)$ we have $C\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)} < 1$. □

Without loss, we can rescale the space, so that $c = 1$. Using Lemma 2, we have that

$$\int_0^\infty -x\big[(g(-x) - g(x)) - (f(-x) - f(x))\big]dx$$

$$>$$

$$\int_0^1 -1 \cdot \big[(g(-x) - g(x)) - (f(-x) - f(x))\big]dx + \int_1^\infty -1 \cdot \big[(g(-x) - g(x)) - (f(-x) - f(x))\big]dx$$

$$=$$

$$(-G_2 + G_3 - G_1 + G_4) - (-F_2 + F_3 - F_1 + F_4) \tag{6}$$

where F_1 through F_4 are the probabilities of according to f on four mutually exclusive and exhaustive intervals

$$\begin{aligned} F_1 &:= \int_1^\infty f(-x)dx = \int_{-\infty}^{-1} f(x)dx \\ F_2 &:= \int_0^1 f(-x)dx = \int_{-1}^0 f(x)dx \\ F_3 &:= \int_0^1 f(x)dx \\ F_4 &:= \int_1^\infty f(x)dx \end{aligned}$$

so $F_1, F_2, F_3, F_4 > 0$ and $F_1 + F_2 + F_3 + F_4 = 1$. G_1 through G_4 are defined analogously. We thus need to show that

$$-G_2 + G_3 - G_1 + G_4 > -F_2 + F_3 - F_1 + F_4. \tag{7}$$

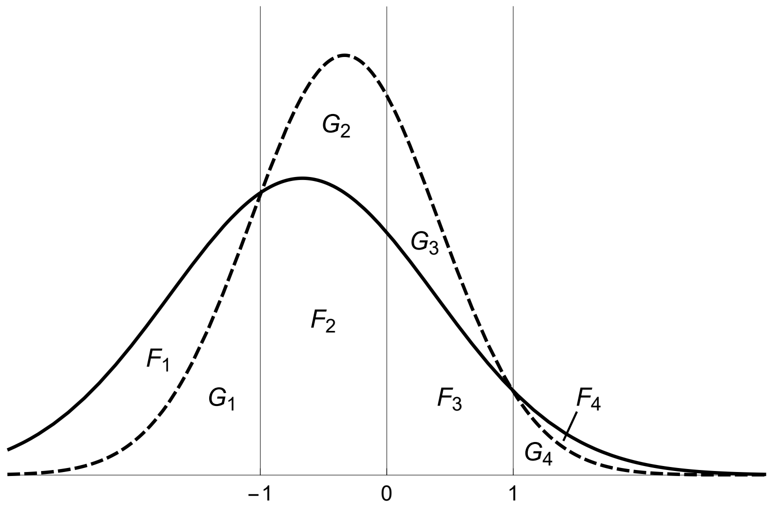

Figure 5: The posteriors f (bold) and g (dashed) after observing $X + \varepsilon = 0$ and $X + \tilde{\varepsilon} = 0$, respectively.

The proof proceeds as follows. As the Figure 5 depicts, G_2 and G_3 are larger than F_2 and F_3, respectively, and G_1 and G_4 are smaller than F_1 and F_4, respectively. We show in Lemma 6 that (7) would hold if G_2 and G_3 were larger than F_2 and F_3, each, by the same factor and similarly for G_1 and G_4. Lemma 3 to 5 argue that in fact these ratios are not the same and thus $-G_2+G_3-G_1+G_4$ is even larger, proving (7).

Before that, we prove the following lemma, which uses the log-concavity and symmetry of the prior density as well as the symmetry of the error density.

Lemma 3. *The posterior probability-ratio* $\frac{f(-x)}{f(x)}$ *is strictly increasing in* x.

Proof. By Bayes' law, f is proportional to the product $f_X(x)f_\varepsilon(s-x) = f_X(x)f_\varepsilon(x)$. The ratio is strictly increasing if its logarithm, which is as follows, is strictly increasing in x. Using the symmetry of f_ε and f_X we obtain

$$\log\left(\frac{f(-x)}{f(x)}\right) = \log\left(\frac{f_X(-x)f_\varepsilon(-x)}{f_X(x)f_\varepsilon(x)}\right) = \log\left(\frac{f_X(-x)}{f_X(x)}\right)$$
$$= \log f_X(-x) - \log f_X(x) = \log f_X(x + 2\mathbb{E}[X]) - \log f_X(x),$$

where we have used that $f_X(x)$ is symmetric around $\mathbb{E}[X] < 0$, so

$$\log f_X(-x) = \log f_X\big(\mathbb{E}[X] + (-x - \mathbb{E}[X])\big) = \log f_X\big(\mathbb{E}[X] - (-x - \mathbb{E}[X])\big) = \log f_X(x + 2\mathbb{E}[X]).$$

By strict concavity of $\log f_X$ and $\mathbb{E}[X] < 0$, the difference $\log f_X(x+2\mathbb{E}[X]) - \log f_X(x)$ is strictly increasing. □

Using Lemma 3, we prove two lemmas regarding ratios of the terms in (6).

Lemma 4. $\frac{F_1}{F_4} \geq \frac{F_2}{F_3}$.

Proof. We have that

$$\frac{F_1}{F_4} = \frac{\int_1^\infty f(-x)dx}{\int_1^\infty f(x)dx} = \frac{\int_1^\infty \frac{f(-x)}{f(x)} f(x)dx}{\int_1^\infty f(x)dx} \geq \frac{\int_0^1 \frac{f(-x)}{f(x)} f(x)dx}{\int_0^1 f(x)dx} = \frac{\int_0^1 f(-x)dx}{\int_0^1 f(x)dx} = \frac{F_2}{F_3}.$$

The inequality holds because by Lemma 3. The two inner terms are the expectation of $\frac{f(-x)}{f(x)}$ with respect to the posterior distribution f conditional on the domain $[1,\infty)$ and $[0,1]$, respectively. The former distribution first-order stochastically dominates the latter, thus the inequality follows from $\frac{f(-x)}{f(x)}$ being strictly increasing in x for $x > 0$. □

The following lemma uses that $\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}$ is decreasing for $x > 0$.

Lemma 5. $1 < \frac{G_2}{F_2} < \frac{G_3}{F_3}$ *and* $\frac{G_1}{F_1} < \frac{G_4}{F_4} < 1$.

Proof. Using $g(x) = \frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)} f(x)C$, where C is the ratio of the integration constants, and $\frac{f_{\tilde{\varepsilon}}(-x)}{f_\varepsilon(-x)} = \frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}$ (by symmetry), we have

$$\frac{G_2}{F_2} = \frac{\int_0^1 \frac{g(-x)}{f(-x)} f(-x)Cdx}{\int_0^1 f(-x)dx} = \frac{\int_0^1 \frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)} Cf(-x)dx}{\int_0^1 f(-x)dx}$$
$$\frac{G_3}{F_3} = \frac{\int_0^1 \frac{g(x)}{f(x)} f(x)dx}{\int_0^1 f(x)dx} = \frac{\int_0^1 \frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)} Cf(x)dx}{\int_0^1 f(x)dx}.$$

Thus, both ratios are expectations of the function $\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}$ over the interval $[0,1]$ multiplied by C but with densities $f(-x)/(\int_0^1 f(-x)dx)$ and $f(x)/(\int_0^1 f(x)dx)$, respectively. Because $\frac{f(-x)}{f(x)}$ is increasing in $x > 0$ by Lemma 3, the former density likelihood-ratio dominates the latter. This implies first-order stochastic dominance, which in turn implies a strictly smaller expectation since $\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}$ is a strictly decreasing function by assumption. Thus, $\frac{G_2}{F_2} < \frac{G_3}{F_3}$.

Moreover, by Lemma 2 and having normalized $c = 1$, $\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}C > 1$ in the interval $[0,1)$. Hence, $\frac{G_2}{F_2}$ and $\frac{G_3}{F_3}$, which are expectations of this ratio, are strictly greater than 1.

The proof of $\frac{G_1}{F_1} < \frac{G_4}{F_4} < 1$ is analogous, but with expectation over the domain $[1,\infty)$. □

Define $\tilde{G}_2 = kF_2$ and $\tilde{G}_3 = kF_3$ with $k > 1$, and $\tilde{G}_1 = lF_1$ and $\tilde{G}_4 = lF_4$ with $l < 1$, such that $\tilde{G}_2 + \tilde{G}_3 = G_2 + G_3$ and $\tilde{G}_1 + \tilde{G}_4 = G_1 + G_4$. By Lemma 5, $\frac{G_3}{F_3} > \frac{\tilde{G}_3}{F_3} = \frac{G_2+G_3}{F_2+F_3} = \frac{\tilde{G}_2}{F_2} > \frac{G_2}{F_2}$, so $Q_3 > \tilde{Q}_3$ and $\tilde{Q}_2 > Q_2$, implying $-G_2 + G_3 > -\tilde{G}_2 + \tilde{G}_3$. Analogously, $-G_1 + G_4 > -\tilde{G}_1 + \tilde{G}_4$.

$$-G_2 + G_3 - G_1 + G_4 > -\tilde{G}_2 + \tilde{G}_3 - \tilde{G}_1 + \tilde{G}_4. \tag{8}$$

Finally, the following lemma concludes the proof.

Lemma 6. $-\tilde{G}_2 + \tilde{G}_3 - \tilde{G}_1 + \tilde{G}_4 > -F_2 + F_3 - F_1 + F_4$.

Proof. Define $r := \frac{F_2}{F_3} > 0$ and $R := \frac{F_1}{F_4} > 0$ where $R > r$ by Lemma 4, and $P := F_2 + F_3$ and $p := F_1 + F_4$.

We have $\tilde{G}_1+\tilde{G}_2+\tilde{G}_3+\tilde{G}_4 = G_1+G_2+G_3+G_4 = F_1+F_2+F_3+F_4 = 1$. Thus, we can define $\Delta := (\tilde{G}_2+\tilde{G}_3)-(F_2+F_3) = (F_1-F_4)-(\tilde{G}_1+\tilde{G}_4)$ with $\Delta > 0$ as well as $kP = P+\Delta$ and $lp = p-\Delta$. From $P = F_2+F_3$ and $r = F_2/F_3$, it follows that $-F_2+F_3 = P(\frac{1}{1+r}-\frac{r}{1+r}) = -P\frac{r-1}{r+1}$ and analogously $-F_1+F_4 = -p\frac{R-1}{R+1}$. So, $(-\tilde{G}_2+\tilde{G}_3)-(-F_2+F_3) = -\Delta\frac{r-1}{r+1}$ and $(-\tilde{G}_1+\tilde{G}_4)-(-F_1+F_4) = \Delta\frac{R-1}{R+1}$. Note that $\frac{d}{dx}\frac{x-1}{x+1} = \frac{2}{(x+1)^2} > 0$ for $x > 0$. Then, by $R > r$, $\frac{R-1}{R+1} > \frac{r-1}{r+1}$, so $-\tilde{G}_2+\tilde{G}_3-\tilde{G}_1+\tilde{G}_4 > -F_2+F_3-F_1+F_4$. □

By (8) and Lemma 6, we obtain $-G_2+G_3-G_1+G_4 > -F_2+F_3-F_1+F_4$. □

6.3 Proof of Lemma 1

Proof. By our definition, the symmetric around 0 random variable $\tilde{\varepsilon}$ is more precise than the symmetric around 0 random variable ε if $[\varepsilon|\varepsilon > 0]$ is smaller in the likelihood ratio order than $[\tilde{\varepsilon}|\tilde{\varepsilon} > 0]$. It is not hard to show that for a non-negative continuous random variables X, aX is smaller in the likelihood ratio order than X for all $0 < a < 1$ if, and only if, $\log f_\varepsilon(e^x)$ is concave for $x > 0$ (e.g. Hu et al., 2004). Applying this result to $X = [\varepsilon|\varepsilon > 0]$ yields the result.

The function $\log f_\varepsilon(e^x)$ is concave for $x > 0$ in particular if f_ε is log-concave and symmetric around 0. Note that

$$\frac{d^2}{dx^2}\log f_\varepsilon(e^x) = \frac{d}{dx}e^x(\log f_\varepsilon)'(e^x) = e^x(\log f_\varepsilon)'(e^x) + e^{2x}(\log f_\varepsilon)''(e^x).$$

The latter term is negative because f_ε is log-concave and the former term is negative because f_ε is also symmetric around 0. □

The main text gives several examples of commonly encountered symmetric, log-concave distributions. Below, we prove for symmetric distributions that are not log-concave that $\log f_\varepsilon(e^x)$ is nevertheless concave.

Non-log-concave examples The *Student-t* distribution with parameter $\nu > 0$, which includes as a special case the Cauchy distribution, gives

$$\begin{aligned} f(x) &\propto \left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2} \\ \Rightarrow \log f(e^x) &= C - \frac{\nu+1}{2}\log(1+\frac{1}{v}e^{2x}) \\ \Rightarrow \frac{d^2}{dx^2}\log f(e^x) &= \frac{d}{dx} - \frac{\nu+1}{2}\frac{\frac{2}{\nu}e^{2x}}{1+\frac{1}{\nu}e^{2x}} = -\frac{\nu+1}{2}\frac{\frac{4}{\nu}e^{2x}}{(1+\frac{1}{\nu}e^{2x})^2} < 0, \end{aligned}$$

and hence has log-concave $f(e^x)$.

Creating a symmetric distribution from the *Pareto* distribution, analogous to the double-exponential distribution, with $\alpha > 0$ gives

$$f(x) \propto x^{-\alpha-1} \Rightarrow \log f(e^x) = C - (\alpha+1)x,$$

with log-linear and hence log-concave $f(e^x)$.

6.4 Proof of Proposition 1

Proof. By symmetry and translation invariance of location experiments, it is without loss to suppose that $X \geq \mathbb{E}[X] = 0$.

First, we show the inequality $\mathbb{E}[X] = 0 \leq \mathbb{E}[\mathbb{E}[X|S]|X]$. We have

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X|S]|X] &= \int_{-\infty}^{\infty} \mathbb{E}[X|S=s] f_\varepsilon(s-X)ds \\ &= \int_{0}^{\infty} \big(\mathbb{E}[X|S=s] f_\varepsilon(s-X) + \mathbb{E}[X|S=-s] f_\varepsilon(-s-X)\big)ds. \qquad (9)\end{aligned}$$

By symmetry of the prior and error densities, $\mathbb{E}[X|S=s] = -\mathbb{E}[X|S=-s]$ and by Fact 1, $\mathbb{E}[X|S=s] > 0$. By symmetry around 0 and quasi-concavity of the error density, $f_\varepsilon(-s-X) = f_\varepsilon(s+X) < f_\varepsilon(s-X)$ and hence, the integrand of (9) is positive for every $s \in [0, \infty)$.

Second, we show $\mathbb{E}[\mathbb{E}[X|S]|X] \leq X$. We have by symmetry of f_ε around 0,

$$\begin{aligned}X &= \int_{-\infty}^{\infty} s f_\varepsilon(s-X)ds \\ &= \int_{0}^{\infty} \big(s f_\varepsilon(s-X) + (-s) f_\varepsilon(-s-X)\big)ds. \qquad (10)\end{aligned}$$

We need to show that (9) is less or equal to (10). By Fact 1, $s \geq \mathbb{E}[X|S=s]$ but $-s \leq \mathbb{E}[X|S=-s]$, preventing a direct conclusion of the result. However, by symmetry, $s - \mathbb{E}[X|S=s] = \mathbb{E}[X|S=-s] - (-s)$ and as before $f_\varepsilon(s-X) > f_\varepsilon(-s-X)$. Thus, for every $s \in [0, \infty)$, the integrand of (9) is less or equal to the integrand of (10). □

6.5 Proof of Proposition 3

We prove Proposition 3. Proposition 2 follows by analogous arguments.

Proof. For the first statement, we can again exploit the symmetry of the conditional expectations $\mathbb{E}_A[X|S]$ and $\mathbb{E}_B[X|S]$ and that the density of S is larger for $x > 0$ than for $x < 0$.

Suppose, without loss, that $X > \mathbb{E}[X] = 0$. By Proposition 1, we have $\mathbb{E}[X] \leq \mathbb{E}[\mathbb{E}_A[X|S]|X]$ and $\mathbb{E}[X] \leq \mathbb{E}[\mathbb{E}_B[X|S]|X]$. It remains to show that $\mathbb{E}[\mathbb{E}_A[X|S]|X] \leq \mathbb{E}[\mathbb{E}_B[X|S]|X]$.

$$\mathbb{E}[\mathbb{E}_A[X|S]|X] = \int_{-\infty}^{\infty} \mathbb{E}_A[X|S=s] f_\varepsilon(s-X)ds = \int_{0}^{\infty} \big(\mathbb{E}_A[X|S=s] f_\varepsilon(s-X) - \mathbb{E}_A[X|S=-s] f_\varepsilon(s-X)\big)ds$$

$$\mathbb{E}[\mathbb{E}_B[X|S]|X] = \int_{-\infty}^{\infty} \mathbb{E}_B[X|S=s] f_\varepsilon(s-X)ds = \int_{0}^{\infty} \big(\mathbb{E}_B[X|S=s] f_\varepsilon(s-X) - \mathbb{E}_B[X|S=-s] f_\varepsilon(s-X)\big)ds$$

By Theorem 1, we have $\mathbb{E}_A[X|S=s] \leq \mathbb{E}_B[X|S=s]$ and $\mathbb{E}_A[X|S=-s] \geq \mathbb{E}_B[X|S=-s]$ for $s > 0$, preventing a direct conclusion of the result. However, by symmetry of the prior and the error density, we know that $\mathbb{E}_B[X|S=s] - \mathbb{E}_B[X|S=s] = \mathbb{E}_A[X|S=-s] - \mathbb{E}_B[X|S=-s]$ and by symmetry and quasi-concavity of the error density and $X > 0$, we have that $f_\varepsilon(s - X) > f_\varepsilon(s + X) = f_\varepsilon(-s - X)$ for $s > 0$. Thus, the integrand of the first equation is smaller than the integrand of the second for any $s > 0$.

For the second statement, conditional on X, the distribution of S is the same for A and B. For any realization S, $\mathbb{E}_A[X|S]$ is closer to $\mathbb{E}[X]$ than $\mathbb{E}_B[X|S]$. So, the conditional distribution of the absolute distance of $\mathbb{E}_A[X|S]$ to $\mathbb{E}[X]$ is smaller in first-order stochastic dominance than the one of $\mathbb{E}_B[X|S]$. □